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ABSTRACT

The general solution to the problem of calculating school enrollments at time (t-1) necessary to meet predetermined graduation or manpower targets at time (t) is derived and examined. The solution, which is obtained by minimizing the squared discrepancy between the given graduations and those predicted by the enrollments, requires the unique generalized inverse of the transformation matrix that maps enrollments into graduations. Information on the rank and order of this matrix is used to evaluate the indeterminancy and approximateness of the solution. (Author)

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BACKCASTING FROM GRADUATION TARGETS TO REQUIRED ENROLLMENTS USING THE GENERALIZED INVERSE OF THE TRANSFORMATION MATRIX

by

Lawrence Bezeau
~~Program in International Education Finance~~
~~University of California~~
~~Berkeley, California 94720~~

NEW ADDRESS

LAWRENCE M. BEZEAU
DEPARTMENT OF EDUCATIONAL ADMINISTRATION
ONTARIO INSTITUTE FOR STUDIES IN EDUCATION
252 BLOOR STREET WEST
TORONTO, CANADA M5S1V6

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BACKCASTING FROM GRADUATION TARGETS TO REQUIRED ENROLLMENTS USING THE GENERALIZED INVERSE OF THE TRANSFORMATION MATRIX

The general solution to the problem of calculating school enrollments at time (t-1) necessary to meet predetermined graduation or manpower targets at time t is derived and examined. It is based on the generalized inverse of the transformation matrix that describes the pupil flow through the school. The author has applied this technique [Jaoua et al, 1974] but at the time there was no solid theoretical basis for it. This paper provides one.

The field of generalized inverses contains a confusion of definitions and names including conditional inverse, pseudoinverse, and weak generalized inverse. The one used here is the stong or unique generalized inverse defined by the four equations below (1 to 4). For many but not all of the findings in this paper a weaker nonunique inverse meeting fewer than 4 of the conditions would have sufficed. To achieve simplicity of presentation and because of the availability of computer software we use only the unique one.

Symbols and conventions are now outlined. "A" represents a general matrix which transforms enrollments into graduations. "G" is its generalized inverse. "B" and "C" are two matrices whose product is A (A = BC) and whose inner dimensions are equal to the rank of all three. "x" is a vector of enrollments and "y" a vector of graduates, each disaggregated by discipline, program or labor force category. "m" and "n" are scalars representing vector or matrix dimensions. "r" always represents the rank of a matrix and sometimes a dimension. Lambdas ("λ") are eigenvalues; "I' is the identity matrix; and "h" is a vector of arbitrary variables with elements h_1, h_2, etc. "Iff" means if and only if. The word "solution" is used rather loosely in that it sometimes refers to an exact genuine solution and sometimes to an approximation. The precise meaning is clear from the context.

The Backcasting Problem

Numerous authors have modeled student flow through school systems [See Johnstone, 1974; and Schiefelbein and Davis, 1974 for reviews] using matrix transformations of the type $Ax = y$ where A_{ij} is the proportion of persons in grade or institution j (often called "state") during the initial time period who move onto state i for a later time period. The vectors x and y represent the number of persons in the various states during the initial and later periods respectively. If the transformation matrix A is square and equal i's and j's represent the same states then the transformation can be compounded over any number of periods by raising A to higher powers. In the case of square A this is similar to a Markov process except that it is frequently seen as deterministic or prescribed rather than stochastic. Furthermore, dropouts are often treated as a residual in which case $[1]^tA \neq [1]^t$ and populations are not conserved.

Frequently educational planners are given this problem in reverse. They are provided with a set of manpower or graduation targets to be met in some future period. They must then calculate the enrollments necessary to bring about the required number of graduations. If the transformation matrix A has equivalent row and column states and is square and nonsingular then the solution is obtained by premultiplying the graduation target vector y by the inverse of A ($x = A^{-1}y$). It is argued below that A is usually not square and that if it is made square by including all states in both the row and column spaces it will, in general, be singular.

Any square transformation matrix that accurately represents progress through a school or school system is nilpotent and therefore singular. Everyone who enters a school eventually leaves either through graduation, dropout, or death. Thus some power of the matrix A which transforms entrants into graduates must map every entrant vector into a zero graduation vector. The only such matrix is the zero matrix. If $A^n = 0$ then A is nilpotent which is sufficient for singularity.

The addition of an absorbing state to the matrix to collect all the graduates prevents nilpotency but does not guarantee invertibility. In general, if the system contains division points where students flow from one stream equally into two or more streams the transformation matrix is singular.

The problem of backcasting from a vector of graduation targets to the prerequisite enrollments requires a more general formulation than that provided by the standard inverse for a nonsingular square matrix. This paper shows that the problem can be solved using the concept of a generalized inverse matrix [Ben-Israel and Greville, 1974; Boullion and Odell, 1971; and Rao and Mitra, 1971] with properties expressed by equations (1) to (4). A is any matrix and G is its generalized inverse.

$$AGA = A \tag{1}$$

$$GAG = G \tag{2}$$

$$AG = (AG)^t \tag{3}$$

$$GA = (GA)^t \tag{4}$$

The generalized inverse G exists for all A and is unique [Graybill, 1969, p. 97.]. Substitution of A^{-1} for G in the above shows that the definition of G simplifies to the standard inverse for invertible A. For rectangular m x n matrices, G has dimensions n x m. The inverse of a zero matrix is the zero matrix with reversed dimensions.

The following two sections develop a "generalized inverse" with the properties needed to solve the backcasting problem. This is then shown to be the matrix that satisfies conditions (1) to (4) above.

The Least Squares Criterion

The starting point is the matrix equation $Ax = y$ with A, a given m x n transformation matrix of rank r; and y, a vector of graduation targets. The solution set which contains values of the enrollment vector x may be empty. If it is, an approximation to x is desired.

The approximation used is least squares, the solution that minimizes the squared discrepancy between the actual y and the y predicted by x. We wish to select an x that minimizes

$$(y-Ax)^t(y-Ax) \tag{5}$$

$$\frac{\partial}{\partial x}(y-Ax)^t(y-Ax) = -2A^t(y-Ax) = 0 \tag{6}$$

$$A^tAx = A^ty \tag{7}$$

A deus ex machina is required at this point because A may not be invertible. This is provided by the singular value decomposition of A [Noble, 1969,

p. 338]. A is decomposed into two matrices B and C of order m x r and r x n respectively such that A = BC. All three matrices are of rank r. For this reason (B^tB) and (CC^t) are invertible. Substitution of BC for A gives:

$$(BC)^t(BC)x = (BC)^ty \tag{8}$$

Premultiplying by C and regrouping gives:

$$(CC^t)(B^tB)Cx = (CC^t)B^ty \tag{9}$$

Because of the invertibility of the leftmost terms the expression reduces to:

$$Cx = (B^tB)^{-1}B^ty \tag{10}$$

Equation (10) gives the least squares solution for x as far as we can take it. It will always exist but may not be unique. There may be an infinite set of x's that reduces the squared discrepancy to zero or to some single positive value.

The Minimum Norm Criterion

To obtain a unique solution another criterion is needed. We choose to minimize the Euclidean norm of x, using the techniques of Lagrange multipliers.

$$L = x^tx + 2\lambda^t(y^* - Cx) \tag{11}$$

$$\text{where } y^* = (B^tB)^{-1}B^ty \tag{12}$$

The symbol y^* defined by equation (12) is used for convenience. The constraint in parentheses in (11) is just equation (10). The term $2\lambda^t$ is a row vector of Lagrange multipliers with the scalar 2, also present for convenience. Partial differentiation yields equations (13) and (14).

$$\frac{\partial L}{\partial x} = 2x - 2C^t\lambda = 0 \qquad (13)$$

$$\frac{\partial L}{\partial \lambda} = 2(y^* - Cx) = 0 \qquad (14)$$

From equations (13) and (14) we obtain (15) and (16) respectively.

$$x = C^t\lambda \qquad (15)$$

$$y^* = Cx \qquad (16)$$

Substituting (15) into (16) gives:

$$y^* = CC^t\lambda \qquad (17)$$

$$\lambda = (CC^t)^{-1}y^* \qquad (18)$$

Substituting (18) into (15) gives

$$x = C^t(CC^t)^{-1}y^* \qquad (19)$$

Finally we substitute (12) into (19) to get rid of y^* and give the least squares minimum norm solution.

$$x = C^t(CC^t)^{-1}(B^tB)^{-1}B^ty \qquad (20)$$

Equation (20) is now used to define G.

$$G = C^t(CC^t)^{-1}(B^tB)^{-1}B^t \qquad (21)$$

Substitution of BC for A and the right side of (21) into equations (1) to (4) will convince the reader that the above expression satisfies the definition of the unique generalized inverse.

Some Generalized Inverse Properties

A number of important results which are used later follow from the definition of G (21) and the fact that A = BC. These are listed below:

$$AG = B(B^tB)^{-1}B^t \qquad (22)$$

$$GA = C^t(CC^t)^{-1}C \qquad (23)$$

$$(GA)^2 = GA \qquad (24)$$

Note that AG and GA are m x m and n x n matrices, respectively. Equation (24), which can be derived by squaring (23), shows that GA is an idempotent matrix. Premultiplication of (24) by $(GA)^{-1}$ demonstrates that the only nonsingular idempotent matrix is the identity matrix. Squaring of (I-GA) followed by the application of (24) can be used to prove that (I-GA) is also idempotent, an important result for the next section. Similarly, (AG) and (I-AG) are idempotent. The expressions for GA and AG can be used to show that GA, AG, (I-GA), and (I-AG) are all symmetric.

The expression for G (21) gives us the least squares minimum norm solution to the equation Ax = y as follows:

$$x = Gy \tag{25}$$

But what is really needed is a solution to equation (10), that is, the set of all x for which the squared discrepancy, $((y-Ax)^t(y-Ax))$, reaches its minimum value. If the minimum squared discrepancy is zero, the solutions are exact; otherwise they are approximate. Either may be unique or indeterminate. The general least squares solution is presented in the following section followed by a proof that it exactly represents the complete solution to equation (10).

The General Least Squares Solution

The complete least squares solution to x is as follows:

$$x = Gy + (I - GA)h \tag{26}$$

where h is an arbitrary n x 1 column vector. This is proven in two steps. Firstly, it is proven that (26) satisfies (10) and secondly, that for every x satisfying (10) there is at least one h such that x also satisfies (26).

The first step begins with the substitution of (26) into (10).

$$C(Gy + (I - GA)h) = (B^tB)^{-1}B^ty \quad (27)$$

Substituting (23) for GA and (21) for G gives the following expression for the left hand side

$$C(C^t(CC^t)^{-1}(B^tB)^{-1}B^ty + (I - C^t(CC^t)^{-1}C)\ h) \quad (28)$$

Removing the outer parentheses and removing $CC^t(CC^t)^{-1}$ in two places results in considerable simplification.

$$(B^tB)^{-1}B^ty + (CI - C)\ h \quad (29)$$

The right term clearly disappears leaving (29) identical to the right hand side of (27). Thus our expression for x, (26), is a solution to (10). We now prove that (26) is the complete solution.

Starting with equation (10) we show that every x that is a solution is also a solution to (26) for some h. Premultiplying (10) by GB gives:

$$GBCx = GB(B^tB)^{-1}B^ty \quad (30)$$

Substituting A for BC in the left and equation (22) in the right gives:

$$GAx = (GAG)y \quad (31)$$

Substituting (2) into the right and rearranging yields:

$$0 = Gy - GAx \quad (32)$$

$$x = Gy + x - GAx \quad (33)$$

$$x = Gy + (I - GA)x \quad (34)$$

Thus any x substituted for h satisfies equations (26) and (10). This does not imply that x can be generated only by itself. The h corresponding to any given x is usually not unique.

In the formula $x = Gy + (I - GA)h$, the first term on the right is the solution corresponding to $h = 0$. It is also the minimum norm solution as shown in (25). The second term provides for all other solutions. Interpreting this as a solution to a system of linear equations, we see that the first term is a particular solution (exact or approximate) of the inhomogeneous system and the second term is the general solution of the homogeneous system. Attention is now concentrated on the second term in order to determine whether the solution is unique, and if not, how many elements there are in the arbitrary vector, h.

The Indeterminacy of the Solution

In this section it is first shown that the rank of GA is equal to the sum of its diagonal elements, called its trace. The proof depends only on GA's symmetry and idempotency so that we can also apply the finding to I and (I-GA). This is followed by three simple proofs, that rank(GA) = rank(A), that rank(I-GA) = n - rank(GA) and finally that rank(I-GA) = nullity(A). We conclude that the columns of (I-GA) form a basis for the null space of A and that the rank of (I-GA) equals the minimum number of elements in the arbitrary vector h.

Since GA is real, square (n x n), and symmetric it is Hermitian, from which a number of important results follow. The eigenvectors of GA are all linearly independent and, in fact, are orthogonal [Noble, p. 321]. GA can be reduced to diagonal form by a transformation that is both a similarity and a unitary transformation.

$$D = Q^{-1}(GA)Q \tag{35}$$

or

$$D = Q^{t}(GA)Q \tag{36}$$

such that,

$$Q^{t} = Q^{-1} \tag{37}$$

Q is an orthonormal matrix and D is an n x n matrix containing the eigenvalues of GA in the diagonal and zeros elsewhere [Noble, 1969, p. 318].

By premultiplying the equation for eigenvalues, $(GA - \lambda I)v = 0$, ty GA and substituting GA for $(GA)^2$ the reader can derive the equation $(1 - \lambda)(GA)v = 0$ and conclude that either $\lambda = 1$ or $GAv = 0$, in which case $\lambda = 0$. Thus the eigenvalues are all zeros and ones.

Consider the relationship between the ranks and eigenvalues of GA and D. The rank of D is equal to the rank of GA because multiplication of a matrix (GA) by nonsingular matrices (Q^{-1} and Q) does not change its rank [Noble, 1969, p. 138]. The rows of the diagonal matrix D can be interchanged to put the identity matrix into the upper left corner and zeros everywhere else. From this it becomes clear that the rank of D is equal to the sum of its eigenvalues. The same conclusion holds for GA since D and GA have the same eigenvalues.

This brings us to an important finding which applies to all symmetric idempotent matrices. Because the sum of the eigenvalues of any square matrix is equal to its trace [Graybill, 1969, p. 223], the rank of GA is equal to its trace. This provides a simple computational technique for determining the rank of GA and (I-GA), both of which are symmetric and idempotent.

Since the trace of a matrix difference is equal to the difference of the traces, the rank of (I-GA) is equal to rank(I) - rank(GA) or n - rank(GA).

It is well known that for any G and A conformable for multiplication $\text{rank}(GA) \leq \text{rank}(A)$ [Noble, 1969, p. 139]. Applying this same principle to equation (1) grouped as $A(GA) = A$, we can conclude that $\text{rank}(GA) \geq \text{rank}(A)$. Therefore, $\text{rank}(GA) = \text{rank}(A)$. By similar reasoning it can be shown that all of A, G, GA, and AG have the same rank, as do B and C.

We can now combine these findings to reach the final conclusion of this section. The general solution contains a matrix (I-GA) whose rank is equal to its trace. Furthermore, its rank is equal to n - rank(GA) or n - rank(A). But n - rank(A) = nullity(A) by Sylvester's Law of Nullity [Gillett, 1970, p. 154]. Therefore, the trace and rank of (I-GA) are equal to the nullity of A. By premultiplying (I-GA)h by A and substituting A for AGA (2) we see that all vectors of the form (I-GA)h are mapped into the zero vector by A. Thus, they are in the null space of A. Because (I-GA) has the rank of the null space of A, (n - rank(A)), it must have exactly (n - rank(A)) linearly independent columns. The columns of (I-GA) therefore form a basis for the null space of A.

Unfortunately, this is not a minimal basis for the null space of A; it contains n vectors but only (n - rank(A)) linearly independent ones. This form of (I-GA) does not provide much insight into the structure of A's null space, a practical and computational problem which is discussed in the section on computation.

The Approximateness of the Solution

This section examines the circumstances under which the equation Ax=y yields exact and approximate values for x. Firstly, an expression for the squared discrepancy that depends only on y and A is derived. This is used to define three possible outcomes for a given A and the set of all y.

The squared discrepancy is $((y-Ax)^t (y-Ax))$ which is minimal when x = Gy + (I-GA)h. Substituting the expression for x in the expression for the squared discrepancy gives:

$$(y-A(Gy + (I-GA)h))^t (y-A(Gy + (I-GA)h)) \tag{38}$$

This expression can be simplified by substituting zero for A(I-GA).

$$(y-Agy)^t (y-AGy) \tag{39}$$

$$y^t y - y^t(AGy) - (AGy)^t y + (AGy)^t AGy \tag{40}$$

This can be simplified by noting that each term is a scalar and, therefore, equal to its transpose and that AG is symmetric and idempotent.

$$y^t(I-AG)y \tag{41}$$

Equation (41) is the expression we seek, the minimum squared discrepancy for a given A and y. Exact solution(s) exist when (41) is equal to zero.

To examine the conditions under which (41) is equal to zero it is necessary to use the fact that a nonnegative quadratic form is equal to zero if and only if the vector valued expression produced by removing the prepositioned vector from the quadratic form is zero.

The proof of this involves the diagonalization of (I-AG) and the transformation of y and y^t into z and z^t by the orthonormal matrix of eigenvectors.

$$y^t(I-AG)y = y^tQ^{-1}DQy \tag{42}$$

We define z as Qy. Since Q is orthonormal $z^t = y^tQ^{-1}$. The expression becomes

$$z^tDz \tag{43}$$

where D is the diagonal matrix of eigenvalues of (I-AG). This expression is a scalar and can be written as follows:

$$z^tDz = z_1(\lambda_1 z_1) + z_2(\lambda_2 z_2) + \ldots \tag{44}$$

where the subscripted z's represent elements of the vector z. Clearly $(z_1(\lambda_1 z_1)) = 0$ if and only if $\lambda_1 z_1 = 0$. Since the eigenvalues are all nonnegative the expression $(z_i\lambda_i z_1)$ must also be nonnegative. Thus, the quadratic form can equal zero only if every term on the right hand side of (44) is zero, which means that every $\lambda_1 z_1 = 0$. This is the key to the proof. It means that:

$$y^tQ^{-1}DQy = 0 \text{ iff } DQy = 0$$

Since Q^{-1} is nonsingular,

$$DQy = 0 \text{ iff } Q^{-1}DQy = 0$$

Combining (45) and (46) and substituting (I-AG) for $Q^{-1}DQ$ completes the proof.

$$y^t(I-AG)y = 0 \text{ iff } (I-AG)y = 0$$

We can now proceed to examine the conditions under which exact solutions exist. There is an exact solution for all y such that

$$y = AGh \tag{48}$$

The first stage of the proof verifies that any y satisfying (48) results in a zero value for (41), the squared discrepancy. Substituting (48) into (41) gives:

$$(AGh)^t(I-AG)AGh \tag{49}$$

Multiplying (49) through by the post positioned AG and application of (1) results in a zero value for the expression. The second stage of the proof shows that an h exists for every y that results in a zero value of (41). This time the expression (I-AG)y from (47) is set to zero, rather than the corresponding quadratic form. Manipulation of this expression quickly shows that:

$$y = AGy \tag{50}$$

This completes the proof. Since AG is generally singular, y is not the only value of h that will generate y.

The results for solution approximateness can be summarized into three cases, the first of which is trivial.

If AG=0 the only y yielding an exact solution is the zero vector. Inspection of (1) shows that AG=0 if and only if A=0.

From equation (41) it is clear that if $AG = I$, there is an exact solution for all y. Since AG is m x m and idempotent this can occur only if the rank of A is m.

The intermediate cases are those for which the rank of AG is greater than zero but less than m. These yield exact solutions for those y that satisfy (50) and approximate solutions for all other values of y.

Computation

This section contains an example of an enrollment backcast and some hints on computation.

The example was solved with a program of 50 Fortran statements that input the transformation matrix (A), the target vector (y), and then called widely available subroutines to do the rest. The subroutines, including one that calculates the unique generalized inverse of a matrix by means of singular value decomposition [Noble, 1969, p. 335-40], are a part of the International Mathematical and Statistical Library [IMSL, 1974]. This library also contains subroutines to do the other matrix and vector operations necessary to calculate and output the complete solution.

*** Place Figures 1 and 2 about here ***

The problem and solution are contained in Figures 1 and 2, respectively. The problem consists of one transformation matrix (A) and two graduation target vectors, y_1, and y_2, which generate two least norm solutions, x_1 and x_2, respectively. The general solution to the homogeneous problem is the same for all y, given A.

The information in Figure 2 permits us to analyze the solutions. The matrix (I-GA) is 4 x 4 with a trace of 2. Its rank is, therefore, 2 and from this GA has a rank of 4-2=2. A and AG are of rank 2. (I-AG) is 5 x 5 and has a rank of 5-2=3. We conclude that there will be an infinite number of least squares solutions, some exact and some approximate depending on y. The expressions for $y^t(I-AG)y$ show that y_1 has exact solutions and y_2 has approximate solutions.

The matrix (I-GA), which is a redundant basis of the null space of A, can be converted to a more convenient form by removing the unnecessary vectors. A systematic way to do this is to reduce it to row echelon form (also known as Hermite normal form) and transpose it. After discarding the zero columns this gives a basis of (n-r) column vectors with the identity matrix in the top (n-r) rows as shown at the bottom of Figure 2.

Two undesirable types of enrollment vectors are often generated. The least serious and most common contains noninteger enrollments that can be rounded. Negative enrollments may occur in either the least norm or the general solution.

The number of enrollment vectors in any solution will be finite if only vectors with nonnegative integer enrollments with a finite maximum sum are considered.

Conclusion

The enrollment backcasting problem Ax=y has been solved. The complete exact or approximate solution is x = Gy + (I-GA)h where G is the generalized inverse of A and h is an arbitrary vector whose minimum dimension is equal to the nullity of A. The solution is exact if $y^t(I-AG)y = 0$; otherwise, it is a least squares approximation. Figure 3 summarizes the rank and order analysis of the solution.

*** Place Figure 3 about here ***

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FIGURE 1

THE EXEMPLARY PROBLEM

$$A = \begin{bmatrix} .040 & .300 & .264 & .136 \\ .210 & .050 & .098 & .162 \\ .440 & .030 & .195 & .335 \\ .110 & .400 & .313 & .197 \\ .200 & .100 & .130 & .170 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} 232.0 \\ 114.0 \\ 430.0 \\ 204.0 \\ 140.0 \end{bmatrix} \qquad y_2 = \begin{bmatrix} 156.0 \\ 124.0 \\ 10.0 \\ 356.0 \\ 354.0 \end{bmatrix}$$

FIGURE 2

THE SOLUTION TO THE EXEMPLARY PROBLEM

$$x_1 = \begin{bmatrix} 163.7931034483 \\ 436.2068965517 \\ 204.4827586207 \\ 215.5172413793 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 7.2150276240 \\ 410.2673758972 \\ 309.1316099753 \\ 136.3507207460 \end{bmatrix}$$

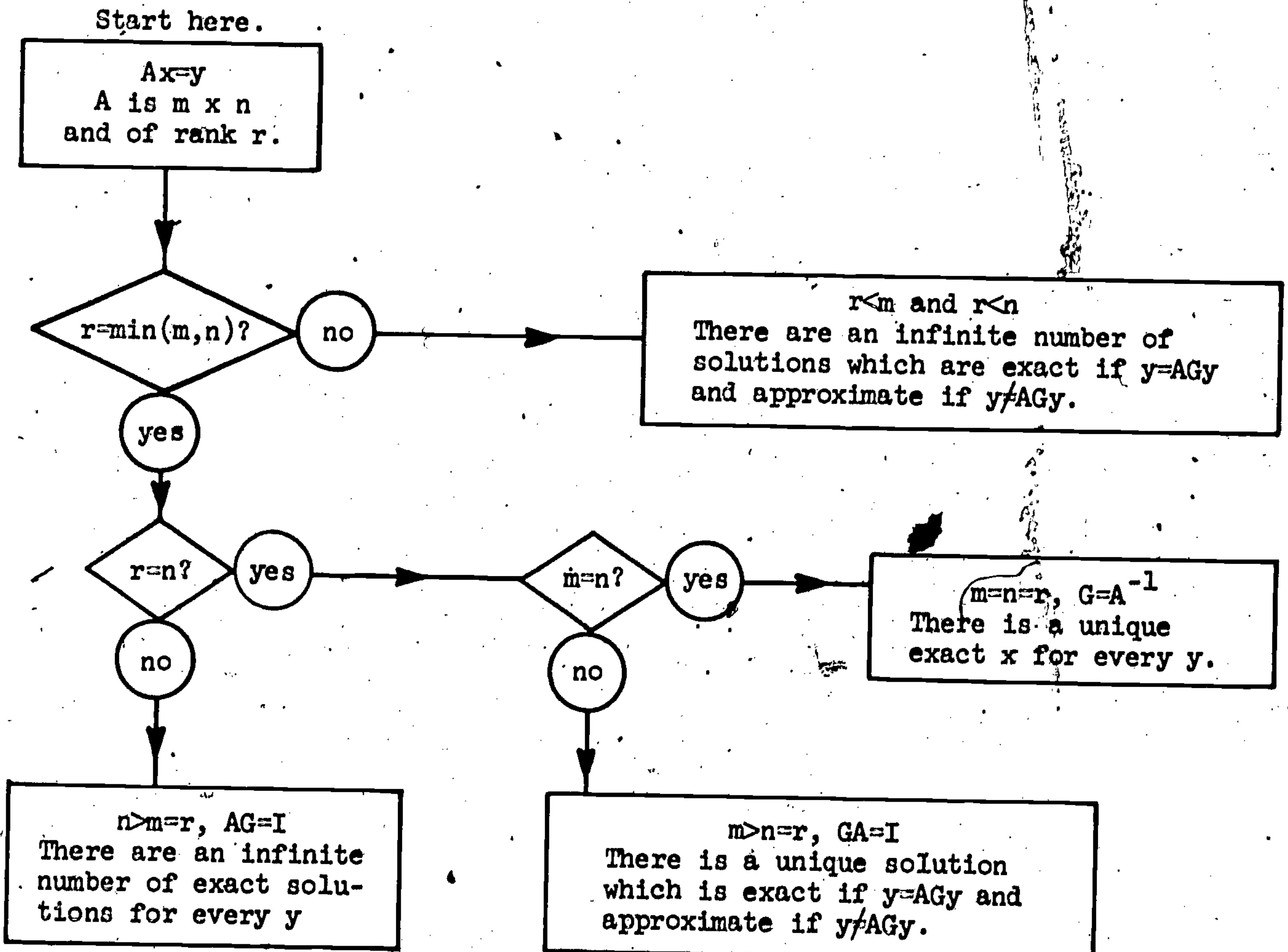
$$y_1^t(I-AG)y_1 = [0.] \qquad y_2^t(I-AG)y_2 = [35344.7]$$

$$(I-GA)h = \begin{bmatrix} .3189655172 & .1810344828 & -.0775862069 & -.4224137931 \\ .1810344828 & .3189655172 & -.4224137931 & -.0775862069 \\ .0775862069 & -.4224137931 & .6810344828 & -.1810344828 \\ -.4224137931 & -.0775862069 & -.1810344828 & .6810344828 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}$$

$$(I-GA)h \text{ reduced} = \begin{bmatrix} 1.0000000000 & 0. \\ 0. & 1.0000000000 \\ .7500000000 & -1.7500000000 \\ -1.7500000000 & .7500000000 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

FIGURE 3

ANALYSIS OF THE GENERAL SOLUTION
BY MEANS OF RANK AND ORDER
INFORMATION



NOTE: "G" is the unique generalized inverse of A.